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ON THE SOLUTION OF OPTIMIZATION PROBLEMS WITH CONSTRAINTS*

S.V. SHAKHVERDIAN

In the development of the method of integral penalty functions methods are proposed for solving optimization problems for dynamic systems with high-order constraints on the phase coordinates. Necessary optimality conditions in the form of a maximum principle are obtained for optimization problems with constraints of different forms on the control rate.

At the present time one of the basic methods for solving optimization problems for dynamic systems with constraints on the phase coordinates is the method of integral penalty functions, which, for example, leads to the adding on of the integral /1—3/

$$K\int_0^T E(g)g^2\,dt,\quad E(g)=\begin{cases}0, & \text{if}\quad g\leqslant 0\\ 1, & \text{if}\quad g>0\end{cases} \tag{0.1}$$

to the performance index in order to take into account inequalities of type $g(x,t)\leqslant 0$. Here x is the system's state vector, K is the penalty coefficient, T is the end of the control period. Note that the optimization problem for dynamic systems with constraints on the phase coordinates has not been solved in the general case by means of introducing integral (0.1). This is connected with the fact that at the instant t_1 a phase trajectory reaches the boundary of the admissible domain the derivative of the integrand in (0.1) with respect to the phase coordinates equals zero, by virtue of which the adjoint variables of the Hamiltonian system are always continuous. Meanwhile it is known that in the general case these variables must undergo discontinuities at instant t_1 /4—6/, in particular, when the order of the constraint $g(x,t)\leqslant 0$ is higher than first. (Here and everywhere below, by the order of a constraint is meant the order of the smallest time derivative of $g(x,t)$ containing the control parameter). In addition, and this is more important, the direct inclusion of $g(x,t)$ in any form in the target functional does not ensure that the phase trajectories lie in the admissible domain in the general case, i.e., the constraint $g(x,t)\leqslant 0$ is not sustained; this will be shown later. If the constraint $g(x,t)\leqslant 0$ is taken into account by adding on the integral

$$\int_0^T \alpha Y\,dt,\quad Y\equiv y^2+g(x,t)=0 \tag{0.2}$$

to the performance index (α is a Lagrange multiplier, y is the additional control parameter), then it can be proved that the method of integral penalty functions is equivalent to the method based on the direct introduction of $g(x,t)$ into the form (0.2), called the multiplier method. (Two methods are said to be equivalent if the solution sets obtained by their use either coincide or are both empty). As a matter of fact, from the extremum of the target functional with respect to y follows $\alpha=0$ on $Q_0=\{t:E(g)=0\}$, $\alpha\neq 0$ on $Q_1=\{t:E(g)=1\}$. Therefore, the equality

$$\int_0^T \alpha Y\,dt=\int_0^T \alpha E(g)g\,dt=0$$

is valid. Consequently

$$\frac{1}{2}K\int_0^T E(g)g^2\,dt=0$$

*Prikl.Matem.Mekhan.,45,No.5,823-832,1981 whence $\alpha = {}^1/_2 Kg$ as $g \to 0$, which proves the equivalence of the two methods. The optimal control problem with a constraint of form $|u^\cdot| \leqslant a$ which is analyzed in the present paper together with others, was studied in /7/, where $u^\cdot$ is the rate of change of the scalar control u and a is a prescribed number. The constraint $|u^\cdot| \leqslant a$ was taken into account in /7/ by introducing u among the phase coordinates and using $u^\cdot$ as the control parameter. The solution obtained in /7/ is incomplete since from it does not follow the solution inside the admissible domain of variation of u and $u^\cdot$. Methods are proposed below for solving the optimization problems for dynamic systems with constraints on the phase coordinates and on the control rate, that are free of the above-mentioned defects.

1. **Problem with phase constraints.** For simplicity of exposition we assume that the phase trajectory reaches the admissible domain's boundary just once and remains on it over the interval $[t_1, t_2]$ and that there is only one pth-order constraint of type $g(x, t) \leqslant 0$. Under these assumptions we consider the standard optimal control: minimize

$$I = \int_0^T f_0(x, u, t)\,dt \tag{1.1}$$

under the constraints

$$x^\cdot = f(x, u, t), \quad x(0) = x^\circ, \quad x(T) = x^T \tag{1.2}$$

$$u(t) \in U = \{u\colon g(u) \leqslant 0\} \tag{1.3}$$

$$x(t) \in R = \{x\colon g(x, t) \leqslant 0\} \tag{1.4}$$

where x is an n-dimensional continuous phase vector having continuous derivatives with respect to t everywhere except at a finite number of points, u is an m-dimensional piecewise-continuous control vector, q is, in the general case, a π-dimensional function on U, continuous and continuously differentiable with respect to u, $\pi \leqslant 2m$, g is a scalar function on $R \times U$, continous and continuously differentiable up to order p with respect to x and u, and f is an n-dimensional vector-valued function. The functions f_0 and f are assumed continuous and continuously differentiable with respect to x and u on $R \times U$.

To prove the necessity of introducing tangential constraints /5/ independently of the method of accounting for $g(x, t) = 0$ on Q_1, as well as the assertion that the direct inclusion of only $g(x, t)$ in any form in the target functional does not ensure constraint (1.4), we consider the method of the extended phase space. It is based on augmenting (1.2) with the system

$$2y_1 y_1^\cdot - y_2 = 0, \quad y_2^\cdot - y_3 = 0, \ldots, \quad y_p^\cdot + g^{(p)}(x, u, t) = 0 \tag{1.5}$$

with right end conditions

$$[y_1^2 + g(x, t)]_T = 0, \ldots, [y_p + g^{(p-1)}(x, t)]_T = 0 \tag{1.6}$$

The solution of system (1.5) with conditions (1.6) has the form /8/

$$y_1^2 + g(x, t) = 0, \quad y_{j+1} + g^{(j)}(x, t) = 0, \quad g^{(j)} \equiv d^j g / dt^j, \quad j = 1, \ldots, p-1 \tag{1.7}$$

From (1.7) we see that if $y_1(t)$ is a real variable everywhere on $[0, T]$, we can ensure the fulfillment of the condition $g(x, t) \leqslant 0$ on $[0, T]$ by adding on system (1.5) with condition (1.6) to system (1.2). However, from (1.7) it does not follow that

$$y_1^2(t) \geqslant 0, \quad \forall t \in [0, T]$$

Consequently, as a result of the introduction of (1.5) with (1.6) the inequality $g(x, t) \leqslant 0$ cannot be observed in the general case.

From (1.5) we see that for the equality $y_1(t) = g(x, t) = 0$ to hold on $[t_1, t_2]$ it is necessary and sufficient to ensure the fulfillment of the following conditions:

$$y_s^\cdot(t) = 0, \quad s \in M = \{1, \ldots, p\}; \quad y_j(t_1) = 0, \quad j = 1, \ldots, p$$

or

$$g^{(s)}(x, u, t) = 0, \quad s \in M, \qquad G(x, t_1) = [g(x, t_1), g^{(1)}(x, t_1), \ldots, g^{(p-1)}(x, t_1)] = 0 \tag{1.8}$$

$$\frac{\partial g^{(s)}(x, u, t)}{\partial u} \begin{cases} = 0, & \text{if } s \in M \setminus p \\ \neq 0, & \text{if } s = p \end{cases}$$

Note that the solving of problem (1.1)—(1.4) with the introduction of system (1.5) and condition (1.6) is essentially equivalent to the solving of problem (1.1)—(1.3) with (0.2), and, consequently, with (0.1). Thus, the methods of integral penalty functions, of multipliers, and of the extended phase space are equivalent, and to take (1.4) into account it is necessary to introduce the second condition of (1.8) into any of them.

Method of integral penalty functions. On the strength of the above the original problem (1.1)—(1.4) must be replaced by the following one: minimize (1.1) under constraints (1.2), (1.3) and (1.8). If the method of integral penalty functions is applied to this problem, then the penalty function must be introduced with the aid of the following equation with boundary conditions

$$\dot{x}_{n+1}=\frac{1}{2}E(g)(g^{(j)})^2\equiv f_{n+1}(x,u,t),\quad j\in M \tag{1.9}$$

$$x_{n+1}(0)=0,\ x_{n+1}(T)=\frac{1}{2}\int_0^T E(g)(g^{(j)})^2\,dt=0 \tag{1.10}$$

where $E(g)$ is defined as in (0.1). It is evident that to account for (1.4) it is necessary to bring into consideration, besides (1.9) with (1.10), the second condition in (1.8), since independently of the means of accounting for $g^{(j)}(x,u,t)=0$ the condition $G(x,t_1)=0$ must always be fulfilled.

We denote

$$x_0^{\cdot}=f_0(x,u,t),\quad x_0(0)=0,\quad x_0(T)=I \tag{1.11}$$

$$\varphi_i=q_i(u)+v_i^2=0,\quad i=1,\ldots,\pi \tag{1.12}$$

where v_i is an additional control parameter enabling us to allow for (1.3) in the form of the equality (1.12). Then problem (1.1)—(1.4) can be stated thus: it is necessary to find vector-valued functions u and v which minimize

$$I^*=I+\lambda G'(x,t) \tag{1.13}$$

under constraints (1.2) and (1.9)—(1.12), where λ is a p-dimensional row vector and G' is a p-dimensional column vector; the prime denotes transposition. The Hamiltonian for this problem is

$$H=\psi^\circ f^\circ+\nu\varphi \tag{1.14}$$
$$f^\circ=(f_0,\ldots,f_{n+1}),\quad \varphi=(\varphi_1,\ldots,\varphi_\pi),\quad \psi^\circ=(\psi_0,\ldots,\psi_{n+1})$$
$$\nu=(\nu_1,\ldots,\nu_\pi)$$

where ψ° is an $n+2$-dimensional adjoint vector and ν is a π-dimensional vector-valued Lagrange multiplier.

The system adjoint to (1.2), (1.9) and (1.1) can be written, using the Hamiltonian, as

$$\psi_0^{\cdot}=0,\quad \psi_i'=-\partial H/\partial x_i,\quad \psi_{n+1}^{\cdot}=0,\quad i=1,\ldots,n \tag{1.15}$$

From the maximum condition for (1.14) we have

$$\frac{\partial H}{\partial u}=\psi^\circ\frac{\partial f^\circ}{\partial u}+\nu\frac{\partial\varphi}{\partial u}=0 \tag{1.16}$$

$$\partial H/\partial v=\nu v=0 \tag{1.17}$$

From (1.17) follows

$$\nu=0,\quad v\neq 0;\quad \nu\neq 0,\quad v=0;\quad \nu=0,\quad v=0 \tag{1.18}$$

Conditions (1.18) are essentially weak, which is due to the reduction of inequality (1.3) to equality (1.12). As a rule, because of the weakness of conditions (1.18) a cycling of the iteration procedure arises; therefore, (1.18) need strengthening. In this connection the validity of the conditions

$$\nu>0,\quad v=0;\quad \nu=0,\quad v\neq 0 \tag{1.19}$$

can be proved by use of the Kuhn—Tucker theorem /9/.

The well-known conjugacy conditions

$$\left(\psi^- -\psi^+ -\lambda\frac{\partial G'}{\partial x}\right)_{t_1}=0,\quad \Phi=\left(H^- - H^+ +\lambda\frac{\partial G'}{\partial t}\right)_{t_1}=0 \tag{1.20}$$

must be fulfilled at instant t_1, where the superscripts plus and minus denote the values of ψ and H from the left and from the right, respectively, of the point t_1. The conjugacy conditions for t_2 are analogous of (1.20) written for $\lambda = 0$. From $x_{n+1}(T) = 0$ and from the continuity and continuous differentiability with respect to $x \in R$ of function g it follows that f_{n+1}, and hence H, are functions that are continuous and continuously differentiable with respect to $x \in R$; also, $g^{(s)} = 0$ on Q_1. It is evident that if we take $g^{(j)} = g$ in (1.9), then the quantity $x_{n+1}(T)$ equals integral (0.1); therefore, the $(n+1)$ st component of ψ° plays the same role in (1.14) as K does in (0.1). Thus, the original problem is reduced to solving the following multipoint boundary-value problem: find $u, v, x, x_{n+1}, \psi^\circ, \nu, \lambda, t_1, t_2$ by simultaneously solving the systems of differential and algebraic Eqs.(1.2), (1.9), (1.12), (1.15) with (1.17) under conditions (1.10), (1.11), (1.19), (1.20) and $G(x, t_1) = 0$. It is assumed here that a nonzero vector-valued function ψ°, $\lambda \neq 0$ and a vector ν satisfying conditions (1.19) exist.

Multiplier method. The method of Lagrange multipliers can be used to solve problem (1.1)-(1.4). For this we select j from set M and we set up this problem's Hamiltonian in the form

$$H = \psi f + \nu\varphi \quad \text{on} \quad Q_0, \quad H = \psi f + \mu_j g^{(j)} + \nu\varphi \quad \text{on} \quad Q_1$$
$$\psi = (\psi_0, \ldots, \psi_n), \quad f = (f_0, \ldots, f_n) \tag{1.21}$$

which on $Q_0 \cup Q_1$ is continuous and continuously differentiable with respect to $x \in R$ since f and $g^{(j)}$ are continuous and continuously differentiable with respect to $x \in R$. Consequently, the vector-valued function ψ can be determined from the system

$$\psi^{\cdot} = -\psi \frac{\partial f}{\partial x_i} \text{ on } Q_0, \quad \psi^{\cdot} = -\left(\psi \frac{\partial f}{\partial x_i} + \mu_j \frac{\partial g^{(j)}}{\partial x_i}\right) \text{ on } Q_1 \tag{1.22}$$
$$\psi^{\cdot} = 0$$

with the use of (1.20). Conditions (1.16)—(1.19) also remain in force here. Problem (1.1)—(1.4) with Hamiltonian H defined by (1.21) and with $j = p$ was considered in /5/.

If we compare the solutions of problem (1.1)—(1.3) and (1.8) for j and $j+1$, we can prove the validity of the conditions

$$\psi_i^{(j)} = \psi_i^{(j+1)} + \mu_{j+1} \frac{\partial g^{(j)}}{\partial x_i}, \quad i = 1, \ldots, n \tag{1.23}$$

$$\mu_j = -\frac{d\mu_{j+1}}{dt}, \quad j = 1, \ldots p-1 \tag{1.24}$$

where $\psi_i^{(j)}$ and $\psi_i^{(j+1)}$ are the solutions of system (1.22) when $g^{(j)}$ and $g^{(j+1)}$, respectively, are introduced into consideration and μ_j and μ_{j+1} are the Lagrange multipliers when $g^{(j)}$ and $g^{(j+1)}$ are in expression (1.21). From the Legendre—Clebsch condition follows

$$(-1)^j \frac{d^j \mu_p}{dt^j} \leqslant 0, \quad j = 1, \ldots, p \tag{1.25}$$

while from the condition of the maximum of (1.21) with respect to u follows

$$\mu_p(t_2) = 0 \tag{1.26}$$

Taking (1.24) and (1.25) into account, we can note that in the general case

$$\mu_j(t_2) \neq 0, \quad \forall j \in M \setminus p$$

The proposed method for taking (1.4) into account, having a number of features in common with the method presented in /4/, differs from it in that the vector G in (1.8) has the dimension $j = p$, while $j \leqslant p$ in /4/, and $Q_0 \neq \phi$ in (1.21), but $Q_0 = \phi$ in /4/.

2. Problems with constraints on the control's rate of variation. We examine three problems with constraints on the control's rate of variation, indirectly related with the problem studied in Sect.1. When solving them we use the well-known method of allowing for two-sided constraints on the control parameter /3,10/, augmented by the requirement of sign-definiteness of the Lagrange multipliers.

Problem 1. Minimize

$$I = \int_0^T f_0(x, u, t)\, dt \tag{2.1}$$

under constraints (1.2), (1.3) and

$$|u_j^{\cdot}| \leqslant a \tag{2.2}$$

We denote

$$u_j^{\cdot} = z \tag{2.3}$$

then (2.2) can be represented as $|z| \leqslant a$ which can be replaced by the equality

$$F = 0, \quad F \equiv a^2 - z^2 - \rho^2 \tag{2.4}$$

with the aid of a parameter ρ. As a result we have to minimize (2.1) with respect to u_s, z, ρ, $s = 1, \ldots, m$, $s \neq j$, under conditions (2.4), (1.2), (1.12). This system's Hamiltonian can be written as

$$H = \psi^\circ f^\circ + \nu\varphi + \mu F$$

where $f_{n+1} \equiv z$, μ is the Lagrange multiplier. Using H, the system adjoint to (1.2), (1.11) and (2.3) can be written as

$$\psi_0^{\cdot} = 0, \quad \psi_i^{\cdot} = -\partial H/\partial x_i, \quad i = 1, \ldots, n \tag{2.5}$$

$$\psi_{n+1}^{\cdot} = -\partial H/\partial u_j \tag{2.6}$$

From the maximum condition for Hamiltonian H, in addition to (1.16) and (1.17) we obtain

$$\partial H/\partial z = \psi_{n+1} - 2\mu z = 0, \quad \partial H/\partial \rho = \mu\rho = 0 \tag{2.7}$$

$$\psi_{n+1} \geqslant 0 \text{ when } z = a, \quad \psi_{n+1} \leqslant 0 \text{ when } z = -a \tag{2.8}$$

From (2.7) and (2.8) follows

$$\mu \geqslant 0, \quad \text{if} \quad |z| = a, \mu = 0, \quad \text{if} \quad -a \leqslant z \leqslant a \tag{2.9}$$

Equations (2.6) and (2.7) are of particular interest in Problem 1. When z is inside the admissible domain during some interval, i.e., $\rho \neq 0, \mu = 0$, then from (2.7) it follows that $\psi_{n+1} = 0$ on this interval, and, consequently, $\psi_{n+1}^{\cdot} = 0$. In this case, from (2.6) we have $\partial H/\partial u_j = 0$, whence u_j is determined by the usual scheme. However, when $|z| = a$, then $\rho = 0, \mu \geqslant 0$ and u_j and μ are determined from (2.3) and (2.7), respectively. Also, if condition (2.9) is fulfilled, then $|z| = a$ is a solution; if not, the solution should be sought within the admissible domain. From (2.7) and (2.9) and from the condition $\psi_{n+1}(\tau) \neq 0$ when $u_j(\tau)$ has been fixed or $\psi_{n+1}(\tau) = 0$, otherwise, $(\tau = 0, T)$, follows $\mu(\tau) \geqslant 0$, $|z(\tau)| = a$ when u has been fixed or $\mu(\tau) = 0$, $-a \leqslant z(\tau) \leqslant a$, otherwise.

Problem 2. Problem 2 differes from Problem 1 in that here we introduce, instead of (2.2), the inequality (b is a prescribed number)

$$\left|\frac{d}{dt}\sum_{j=1}^{m} u_j\right| \leqslant b \tag{2.10}$$

We denote

$$V = \sum_{j=1}^{m} u_j, \quad \varphi_0 = V - \sum_{j=1}^{m} u_j = 0, \quad V^{\cdot} = \beta \tag{2.11}$$

The last equation in (2.11) is added on to system (1.2), thanks to which, as well as to (1.11), the dimension of the phase space becomes equal to $n + 2$; also, for V we take the left and right ends to be free. Then constraint (2.10) can be presented as: $|\beta| \leqslant b$ or $F_1 = b^2 - \beta^2 - \gamma^2 = 0$, where γ is the additional control parameter. This problem's Hamiltonian is (μ_1 is a Lagrange multiplier)

$$\begin{gathered} H = \psi^\circ f^\circ + \nu^\circ \varphi^\circ + \mu_1 F_1 \\ f_{n+1} \equiv \beta, \quad \varphi^\circ = (\varphi_0, \varphi), \quad \nu^\circ = (\nu_0, \nu) \end{gathered} \tag{2.12}$$

For Problem 2 the adjoint equations of the Hamiltonian system are

$$\psi_0^{\cdot} = 0, \quad \psi_i^{\cdot} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n; \quad \psi_{n+1} = -\frac{\partial H}{\partial V} \tag{2.13}$$

From the maximum condition for (2.12), together with (1.16) and (1.17), we have

$$\frac{\partial H}{\partial \beta}=\psi_{n+1}-2\mu_1\beta=0, \quad \frac{\partial H}{\partial \gamma}=\mu_1\gamma=0 \tag{2.14}$$

$$\psi_{n+1}\geqslant 0 \quad \text{when} \quad \beta=b; \quad \psi_{n+1}\leqslant 0 \quad \text{when} \quad \beta=-b \tag{2.15}$$

From (2.14) and (2.15) we obtain $\mu_1 \geqslant 0$, if $|\beta|=b$; $\mu_1=0$, if $-b\leqslant\beta\leqslant b$. When $\mu_1=0$, β is found inside the admissible domain, while from (2.14) and the last equation in (2.13) we have

$$\psi_{n+1}=0, \quad \partial H/\partial V=0, \quad \nu_0=0$$

However, if $\mu_1 \geqslant 0$, $|\beta|=b$; then $\nu_0\neq 0$, and ν_0 is determined from the condition that the equality $\varphi_0=0$ is fulfilled, in which V is computed from the last equation of (2.11). Note that in this problem $\psi_{n+1}(0)=\psi_{n+1}(T)=0$ since the variable V can be examined with free ends.

Problem 3. In this problem, instead of inequality (2.2) (Problem 1), we examine the equality ($l(t)$ is a prescribed function of time)

$$\frac{d}{dt}\sum_{j=1}^{\tau}u_j=l(t), \quad \tau\leqslant m \tag{2.16}$$

We denote

$$\varphi_0=V-\sum_{j=1}^{\tau}u_j$$

Then (2.16) takes the form

$$V^{\cdot}=l(t) \tag{2.17}$$

Equation (2.17) is added on to system (1.2) and $\varphi_0=0$ is introduced as an additional constraint. Here too the variable V is introduced with free ends. On the strength of this the Hamiltonian for Problem 3 can be written thus

$$H=\psi^\circ f^\circ+\nu^\circ\varphi^\circ \tag{2.18}$$
$$(f_{n+1}\equiv l(t), \quad \varphi^\circ=(\varphi_0,\varphi), \quad \nu^\circ=(\nu_0,\nu))$$

For this problem the necessary conditions for the maximum of H are analogous to Eqs.(1.16) and (1.17), while the adjoint system is analogous to system (2.13), and

$$\psi_{n+1}^{\cdot}=-\frac{\partial H}{\partial V}=-\nu_0, \quad \psi_{n+1}(0)=\psi_{n+1}(T)=0$$

We remark that the constant of integration, obtained when solving Eq.(2.17), must be determined from the condition $\psi_{n+1}(T)=0$. Note that constraint (2.16) can be accounted for in the same way. From (2.17) we obtain

$$V=L(t)+C, \quad L(t)\equiv\int_0^t l(t)\,dt$$

where the integration constant C is a parameter in this case, thanks to which Problem 3 is reduced to a parametric problem.

3. Example. Find the control $u(t)$ minimizing the functional

$$I=\frac{1}{2}\int_0^1 u^2\,dt \tag{3.1}$$

under the constraints

$$x_1^{\cdot}=x_2, \; x_2^{\cdot}=u, \; x^\circ=(0,1), \; x^T=(0,-1) \tag{3.2}$$

$$g(x_1)=x_1(t)-d\leqslant 0, \; d<1/_6 \tag{3.3}$$

This example is taken from /5/ where the exact solution was obtained by introducing $G(x,t_1)=0$ and $g^{(p)}(u)=0$ on $[t_1,t_2]$, where $p=2$ for constraint (3.3). This simple example is of interest because by it we can easily show that the usual method of integral penalty functions, as well as other methods based on the direct inclusion of $g(x,t)=0$ in any form in the target functional, for taking (3.3) into account, are not suitable. As a matter of fact, if constraint (3.3) is taken into account, say, by using the equality

$$\varphi=y^2+g(x_1)=0 \tag{3.4}$$

then the Hamiltonian for the example being examined can be written as

$$H = \frac{1}{2}\psi_0 u^2 + \psi_1 x_2 + \psi_2 u + \lambda\varphi \tag{3.5}$$

while the adjoint system has the form (y is the additional control parameter)

$$\psi_0 = -1, \quad \psi_1^{\cdot} = -\lambda\frac{\partial\varphi}{\partial x_1}, \quad \psi_2^{\cdot} = -\psi_1 \tag{3.6}$$

From the maximum condition for (3.5) we obtain

$$\partial H/\partial u = \psi_2 - u = 0,\ \psi_2 = u \tag{3.7}$$

$$\partial H/\partial y = \lambda y = 0 \tag{3.8}$$

From (3.8) follows

$$\lambda \neq 0,\ y = 0;\quad \lambda = 0,\ y \neq 0;\quad \lambda = 0,\ y = 0 \tag{3.9}$$

Since $y \neq 0$ on $[0, t_1]$ and $[t_2, T]$, $\lambda = 0$; consequently, $\psi_1 = \text{const}$ on these intervals. However, $y = 0$ on $[t_1, t_2]$, i.e., $g(x_1) = 0, x_1 = d, x_1^{\cdot} = x_2 = 0, x_2^{\cdot} = u = 0$. But when $y = 0$, according to (3.9), $\lambda \neq 0$ or $\lambda = 0$. If $\lambda \neq 0$, then from (3.6) it follows that $\psi_2 \neq 0$ on $[t_1, t_2]$; consequently, $u \neq 0$, $x_2 \neq 0$ and $x_1 \neq d$. In other words, condition (3.3) cannot be ensured for any nonzero λ. However, if $\lambda = 0$, then $\psi_1 = \text{const}$ on $[0, T]$ and ψ_2 in the general case is a linear time function, which again does not ensure the fulfillment of (3.3).

We remark that problem (3.1)—(3.3) was solved by use of penalty functions in the usual manner, i.e., without the introduction of condition (1.8). When solved thus the following picture is observed. As the penalty function's coefficient increased, the phase trajectory contracted to the boundary of the admissible domain, while the right end in the phase trajectory essentially receded from the point $x^T = (0, -1)$. However, when the conditions on the right end were retained, constraint (3.3) was not fulfilled for any coefficients of the penalty functions. All of this again confirms the necessity of introducing condition (1.8).

For the example at hand condition (1.8) has the form

$$G' = \left\| \begin{matrix} x_1 - d \\ x_2 \end{matrix} \right\|_{t_1} = 0 \tag{3.10}$$

$$g^{(j)} = 0,\ t \in [t_1, t_2],\ j \in M = \{1, 2\}$$

In problem (3.1)—(3.3) constraint (3.3) should be taken into account, as noted above, by introducing into consideration (3.10) and either $g^{(1)} = 0$ or $g^{(2)} = 0$. Problem (3.1)—(3.3) was studied in /5/ with the aid of (3.10) and $g^{(2)} = 0$; therefore, here we give the solution with the introduction of the condition $g^{(1)} = 0$ on the interval $[t_1, t_2]$. For problem (3.1)—(3.3), when (3.3) is accounted for by using (3.10) and $g^{(1)} = x_2 = 0$, the Hamiltonian can be written as

$$H = \frac{1}{2}\psi_0 u^2 + \psi_1 x_2 + \psi_2 u + \mu_1 E(g)\, g^{(1)} \tag{3.11}$$

The adjoint system is

$$\psi_0 = -1,\ \psi_1 = \text{const},\ \psi_2^{\cdot} = -(\psi_1 + \mu_1 E(g))$$

Condition (3.7) remains in force when H is defined by (3.11). With due regard to the conjugacy conditions (1.20) and (1.21) we obtain the following solution of problem (3.1)—(3.3)

$$\mu_1 = -\psi_1 \quad \text{on} \quad [t_1, t_2]$$

$$\psi_1 = \begin{cases} \dfrac{2}{9d^2}, & t \in [0, 3d] \\ -\dfrac{2}{9d^2}, & t \in [3d, 1] \end{cases}$$

$$\psi_2 = \begin{cases} \dfrac{2}{3d}\left(1 - \dfrac{t}{3d}\right), & t \in [0, 3d] \\ 0, & t \in [3d, 1 - 3d] \\ \dfrac{2}{3d}\left(1 - \dfrac{1-t}{3d}\right), & t \in [1 - 3d, 1] \end{cases}$$

The expressions for x_1 and x_2 are analogous to those obtained in /5/.

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